Section 2.3. Linear Equations (1st order)

Def: A FOLE is an equation of the form:

$$\underbrace{y' + P(x)\,y}_{LHS} = f(x) \quad \text{(standard form)} \quad ①$$

⊕ Not separable in general

⊕ Method of solving:

Idea: LHS is the derivative of some product (product rule)

Product rule: $(uv)' = u'v + uv'$ ↪ compare LHS

$y = u$, $v = \mu$

Look for an integrating factor $\mu(x)$

$$\mu(x)\,y' + \mu(x)\,P(x)\,y = \mu(x)\,f(x)$$

Need $\mu(x)\,P(x) = v' = \mu'(x)$

$$\Leftrightarrow \int \frac{\mu'(x)}{\mu(x)} = \int P(x) \quad ②$$

② determines the integrating factor

$$\ln(\mu(x)) = \int P(x)\,dx$$

$$\boxed{\mu(x) = e^{\int P(x)\,dx}}$$

Once $\mu(x)$ is determined then the eq becomes

$$(\mu(x)\,y)' = \mu(x)\,f(x) \quad ③$$

Taking integration on both sides of (3)

$$\mu(x)\, y = \int \mu(x) f(x)\, dx$$

$$\boxed{y = \frac{\int \mu(x) f(x)\, dx}{\mu(x)}}$$

In summary : $y' + P(x)\, y = f(x)$ (1)

$\mu(x) = e^{\int P(x)\, dx}$: integrating factor

$\int (\mu y)' = \int \mu f$ (3)

$$y = \frac{\int \mu(x) f(x)\, dx}{\mu(x)}.$$

Note : Homogeneous FOLE : $(f(x) \equiv 0)$

$\int \mu(x)\, 0\, dx = C$ = constant

$$y = \frac{C}{\mu(x)}.$$

Ex : (1) $\boxed{xy' + y = 0}$ (Not in standard form)

- Transform into standard form :

$\boxed{y' + y \cdot \frac{1}{x} = 0}$ (in standard)

- $P(x) = \frac{1}{x}$, $\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$$- \int (yx)' = x \cdot 0 = \int 0$$

$$yx = c \Rightarrow \boxed{y = \frac{c}{x}} \quad x \neq 0$$

is the general sol.

- Consider $x = 0$, $y = 0$ $\Rightarrow$ Trivial sol

② $\boxed{y' + \frac{y}{x} = 4x^2}$ FOLE.

$$P(x) = \frac{1}{x}, \quad \mu(x) = e^{\int \frac{1}{x} dx} = x$$

$$\int (xy)' = \mu(x) f(x) = x(4x^2) = \int 4x^3\, dx$$

$$xy = x^4 + C$$

$$\boxed{y = \frac{x^4 + C}{x} = x^3 + \frac{C}{x}}$$

general sol

non-h

Observation: Sol to $(y' + \frac{y}{x} = 4x^2)$ is $x^3 + \frac{C}{x}$

sol to $(y' + \frac{y}{x} = 0)$ is $\frac{C}{x}$

homogeneous

Principle: The sol to a non-homogeneous eq is the sum of 2 solutions $y = y_c + y_p$

y_c = complementary sol = sol to the

corresponding homogeneous eq

y_p = particular sol

Ex : $y' + 2y = \boxed{\begin{cases} 2 & 0 \le x \le 3 \\ 0 & x > 3 \end{cases}}$, $\boxed{y(0) = 0}$ IVP

⊕ $\boxed{0 \le x \le 3}$, $y' + 2y = 2$, $y(0) = 0$

$P(x) = 2$, $\mu(x) = e^{\int 2\,dx} = e^{2x}$

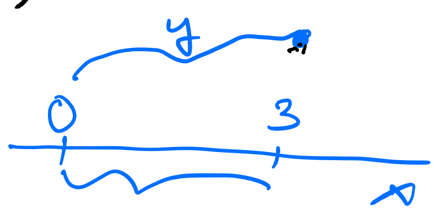
$$\int (\mu y)' = \int \mu(x)\, 2 = \int 2 e^{2x}\,dx$$

$$\mu y = e^{2x} + C$$

$$y = \frac{e^{2x} + C}{e^{2x}} = 1 + \frac{C}{e^{2x}} \Leftarrow \text{general sol}$$

$$y(0) = 0 \Rightarrow 1 + \frac{C}{e^0} = 0, \quad 1 + C = 0 \Rightarrow C = -1.$$

$$y = 1 - \frac{1}{e^{2x}}, \quad y(3) = 1 - \frac{1}{e^6}$$



⊕ $x > 3$, $y' + 2y = 0$

$$y = \frac{C}{e^{2x}} \Leftarrow \text{has to determine } C \text{ (IVP)}$$

$$y(3) = 1 - \frac{1}{e^6} = \frac{C}{e^6}$$

$$1 = \frac{C+1}{e^6} \Rightarrow C + 1 = e^6 \Rightarrow C = e^6 - 1.$$

The final answer $y = \begin{cases} 1 - \dfrac{1}{e^{2x}}, & 0 \le x \le 3 \\ \dfrac{e^6 - 1}{e^{2x}} & x > 3 \end{cases}$

2.4 Exact Equations

Theoretical discussion:

Differential of a function of 2 variables $F = F(x, y)$

$$\boxed{0 = dF} = \underbrace{M\,dx + N\,dy}, \quad \boxed{M = \frac{\partial F}{\partial x}, \quad N = \frac{\partial F}{\partial y}}$$

$$N\frac{dy}{dx} + M = N y' + M$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial N}{\partial x}}$$

F

x

$y = y(x)$

Def: The expression $M(x,y)\,dx + N(x,y)\,dy$ is exact if it corresponds to the differential of some function $F(x,y)$.

A first order ODE of the form $\underbrace{M\,dx + N\,dy}_{LHS} = 0$ is exact if the LHS is exact.

In practice: Criterion for Being exact

$M\,dx + N\,dy$ is exact iff $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$.

Ex: $x\,dx + y\,dy = 0 \Rightarrow$ exact.

$M(x,y) = x, \quad N(x,y) = y$

$\frac{\partial M}{\partial y} = 0, \quad \frac{\partial N}{\partial x} = 0$

Ex: $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0 \Rightarrow$ non-exact

$M = xy, \quad N = 2x^2 + 3y^2 - 20$

$\frac{\partial M}{\partial y} = x \neq \frac{\partial N}{\partial x} = 4x$

Method of solving.

Case 1: Exact $\quad M\,dx + N\,dy = 0$

$F =$ constant, $\boxed{M = \frac{\partial F}{\partial x}, \ N = \frac{\partial F}{\partial y}}$

Need to determine F!

$M = \frac{\partial F}{\partial x} \Rightarrow F(x,y) = \underline{\int M(x,y)\,dx} + g(y)$

$N = \frac{\partial F}{\partial y}, \quad N(x,y) = \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + g'$

$g' = N - \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right)$

Solve for $g(y) \Rightarrow F(x,y)$

Ex: $x\,dx + y\,dy = 0$

$M = x, \quad N = y,$

$\frac{\partial F}{\partial x} = M, \quad F = \int M\,dx + g(y) = \int x\,dx + g(y)$
$= \frac{x^2}{2} + \underline{g(y)}$

$\frac{\partial F}{\partial y} = N, \quad y = N = \frac{\partial}{\partial y}\left(\frac{x^2}{2}\right) + g' = g'$

$g'(y) = y \Rightarrow g = \int y\,dy = \frac{y^2}{2} + C.$

$$F = \underbrace{\frac{x^2}{2} + \frac{y^2}{2}}_{} + C$$

The <u>implicit solution</u> to the ODE is
$F = \text{constant}$

$$\Leftrightarrow \underbrace{\frac{x^2}{2} + \frac{y^2}{2}}_{} = C$$

<u>Case 2</u>: Non-exact to exact via integrating factor

$\tilde{M}\,dx + \tilde{N}\,dy = 0, \quad \frac{\partial \tilde{M}}{\partial y} \neq \frac{\partial \tilde{N}}{\partial x}$

$(\mu \tilde{M})\,dx + (\mu \tilde{N})\,dy = 0$

$M = \mu \tilde{M}, \quad N = \mu \tilde{N}$

To be exact: $\mu_y \tilde{M} + \mu \tilde{M}_y - \mu_x \tilde{N} - \mu \tilde{N}_x = 0$

In general, not possible.

Special cases:

⊕ $\frac{\tilde{M}_y - \tilde{N}_x}{\tilde{N}} = f(x)$ (not depending on y)
then $\mu = e^{\int f(x)\,dx}$

⊕ $\frac{\tilde{M}_y - \tilde{N}_x}{\tilde{M}} = g(y)$ (not depending on x)
then $\mu = e^{-\int g(y)\,dy}$

<u>Ex</u>: $xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$

$\tilde{M} = xy, \quad \tilde{N} = 2x^2 + 4y^2 - 20$

$\tilde{M}_y = \frac{\partial \tilde{M}}{\partial y} = x \neq \frac{\partial \tilde{N}}{\partial x} = 4x = \tilde{N}_x$

Observe: $\frac{\tilde{M}_y - \tilde{N}_x}{\tilde{M}} = \frac{x - 4x}{xy} = \frac{-3}{y} = g(y)$

$\mu = e^{\int \frac{3}{y}\,dy} = e^{3\ln y} = y^3$

$y^3 (xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0)$

$xy^4\,dx + y^3(2x^2 + 3y^2 - 20)\,dy = 0$

$M = xy^4, \quad N = y^3(2x^2 + 3y^2 - 20)$

$\frac{\partial M}{\partial y} = 4xy^3, \quad \frac{\partial N}{\partial x} = y^3 4x$ ✓ exact!

Using the method for exact equation:

$F(x,y) = \int M\,dx + g(y) = \int xy^4\,dx + g(y)$

$= y^4\frac{x^2}{2} + g(y)$

$N = \frac{\partial F}{\partial y} \Rightarrow y^3(2x^2 + 3y^2 - 20) = \frac{\partial}{\partial y}\left(\frac{y^4x^2}{2}\right) + g'(y)$

$$= 2y^3x^2 + g'(y)$$

$$\Rightarrow g'(y) = y^3(3y^2-20) = 3y^5 - 20y^3$$

$$\Rightarrow g(y) = \int g'(y)dy = \frac{y^6}{2} - 5y^4$$

$$F(x,y) = \frac{x^2y^4}{2} + \frac{y^6}{2} - 5y^4$$

General implicit solution $\frac{x^2y^4}{2} + \frac{y^6}{2} - 5y^4 = C$

Ex: $(2x + yx^{-1})\,dx + (xy-1)\,dy = 0$

$\tilde{M} = 2x + yx^{-1}$, $\tilde{N} = xy - 1$

$\tilde{M}_y = x^{-1}$, $\tilde{N}_x = y$ $\Rightarrow$ Non-exact

Observe: $\frac{\tilde{M}_y - \tilde{N}_x}{N} = \frac{x^{-1} - y}{xy-1} = -\frac{1}{x} = f(x)$

$$\mu = e^{\int -\frac{1}{x}dx} = x^{-1}$$

$\Rightarrow (2 + yx^{-2})\,dx + (y - x^{-1})\,dy = 0$ Exact!